Induction by Direct Inference Meets the Goodman Problem

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Abstract

I here aim to show that a particular approach to the problem of induction, which I will call "induction by direct inference", comfortably handles Goodman's problem of induction. I begin the article by describing induction by direct inference. After introducing induction by direct inference, I briefly introduce the Goodman problem, and explain why it is, *prima facie*, an obstacle to the proposed approach. I then show how one may address the Goodman problem, assuming one adopts induction by direct inference as an approach to the problem of induction. In particular, I show that a relatively standard treatment of what some have called the "Reference Class Problem" addresses the Goodman Problem. Indeed, plausible and relatively standard principles of direct inference yield the conclusion that the Goodman inference (involving the grue predicate) is defeated, so it is unnecessary to invoke considerations of 'projectibility' in order to address the Goodman problem. I conclude the article by discussing the generality of the proposed approach, in dealing with variants of Goodman's example.

Keywords: *Problem of induction, New riddle of induction, Reference class problem, Direct inference*

The problem of induction consists of the difficulty of explaining why agents are justified in believing the conclusions of inductive inferences, and, in particular, of explaining why they are justified in believing the conclusions of those inductive inferences that are correct.[1] There is of course much room for debate about the sort of factors that would be sufficient for justifying such beliefs, and hence about what would count as an adequate explanation of the justification of such beliefs. It is in this area where one may raise objections to the treatment of the problem of induction proposed here. I will not attempt to address such objections,

Kriterion – Journal of Philosophy, 2018, 32(2): 1–24.
http://www.kriterion-journal-of-philosophy.org

but see, for example, (Stove 1986) and (McGrew 2001). My starting point is rather to show that a form of inference that I will call "statistical induction" can be, in a certain manner, reduced to direct inference. I will then proceed upon the assumption that direct inference is relatively unproblematic, so that tracing the justificatory basis of statistical induction to direct inference yields a possible means of addressing the problem of induction. The remaining task, at that point, is to illustrate how we may thereby dispense with the Goodman problem. In the present section, I sketch the nature of statistical induction and direct inference, and show that the former form of inference is, in some sense, reducible to the latter.

An account of statistical induction codifies and explains the justificatory basis of inferences that move from a premise, describing the relative frequency of some characteristic among a sample, to a conclusion that states that the incidence of the chosen characteristic among a respective population (from which the sample was drawn) is likely to be very similar to its incidence among the sample. In order to describe such inferences, I use the notation "PROB" to refer to a probability function that takes propositions as arguments, and is understood as designating the (potentially imprecise) personal probabilities (or degrees of belief) that an agent ought to adopt given the evidence the agent has. So the injunction to *infer* or *believe* a given personal probability statement is tantamount to the injunction to adopt the personal probability that one ought to have, given one's evidence. I use the notation "freq" (for "frequency") to refer to a function that takes two sets as arguments, and returns the relative frequency of the first set among the second. So "freq[G|F] = 0.5" expresses that the relative frequency of Fs that are Gs is 0.5. Given this notation, instances of statistical induction exemplify the following defeasible inference schema (where S is the observed *sample* of Fs):[2]

(1) From $S \subseteq F$ and $\text{freq}[G|S] = r$ infer that $\text{PROB}(\ \text{freq}[G|F] \approx r\ )$ is high.[3]

An underappreciated fact about statistical induction is its 'reducibility' to so called "direct inference" (cf. Williams 1947; Kyburg 1956, 1961, 1974; Stove 1986; McGrew 2001; Thorn 2014). A direct inference proceeds from a premise stating that the frequency with which members of a given reference class, R, are members of a respective target class, T, is r, and a premise stating that a given object, c, is an element of R, and yields the conclusion that the probability that c is a member of T is r.[4] Typical instances of direct inference satisfy the following schema:

(2) From c $\in$ R and freq[T|R] $= r$ infer that PROB(c $\in$ T) $= r$.

The 'reduction' of statistical induction to direct inference proceeds from theorems of mathematics which describe the frequency with which subsets of a set (approximately) agree with the set (of which they are a subset) regarding the frequency of any characteristic. There are numerous mathematical theorems that capture the proceeding fact, including:[5]

(3) *Theorem 1:* $\forall\varepsilon,\delta{>}0\text{: } \exists n\text{: } \forall\mathbb{F},\mathbb{G}\text{: } n{<}|\mathbb{F}|{<}|\omega| \Rightarrow$
$$\text{freq}[\{x\text{: freq}[\mathbb{G}|x] \approx_\varepsilon \text{freq}[\mathbb{G}|\mathbb{F}]\} \mid \{x\text{: } x{\subseteq}\mathbb{F}\}] > 1{-}\delta.$$[6]

(4) *Theorem 2:* $\forall\mathbb{F},\mathbb{G}\text{: } \forall\varepsilon,\delta,n{>}0\text{: } n{\geq}1/(4\varepsilon^2\delta) \Rightarrow$
$$\text{freq}[\{x\text{: freq}[\mathbb{G}|x] \approx_\varepsilon \text{freq}[\mathbb{G}|\mathbb{F}]\} \mid \{x\text{: } x{\subseteq}\mathbb{F} \wedge |x|{=}n\}] > 1{-}\delta.$$

Theorem 1 tells us that if $\mathbb{F}$ is sufficiently large (i.e., $|\mathbb{F}| > n$, for a sufficiently large n), then we can be certain that the frequency with which subsets of $\mathbb{F}$ are in (at least) approximate agreement with $\mathbb{F}$ (by a margin of ε) regarding the frequency of any given characteristic $\mathbb{G}$ is at least $1{-}\delta$. Theorem 2 concerns subsets of $\mathbb{F}$ whose size is at least n. The theorem tells us that if n is at least $1/(4\varepsilon^2\delta)$, then the frequency with which n membered subsets of $\mathbb{F}$ are in (at least) approximate agreement with $\mathbb{F}$ (by a margin of ε) regarding the frequency of any given characteristic $\mathbb{G}$ is at least $1{-}\delta$. For example, Theorem 2 tells us that the frequency with which 2,000 element subsets (or samples) of a set (or population) are in approximate agreement with the set (by a margin of 0.05) regarding the frequency of any characteristic, $\mathbb{G}$, is at least 0.95. While Theorem 2 is the most useful in practice (since it illustrates how to compute precise values for δ, given values for n and ε), Theorem 1 offers a fairly simple illustration of how direct inference can be used to emulate statistical induction. In particular, Theorem 1 is applicable in generating the major premises for direct inferences of the following form (omitting mention of

ε and δ, which are assumed to be very small, inasmuch as |F| is assumed to be very large):

(5) From S $\in$ {x: $x$$\subseteq$F} &
freq[{x: freq[G|x] $\approx$ freq[G|F]} | {x: $x$$\subseteq$F}] $\approx$ 1
infer that PROB(S $\in$ {x: freq[G|x] $\approx$ freq[G|F]}) $\approx$ 1
(i.e., that PROB(freq[G|S] $\approx$ freq[G|F]) $\approx$ 1).

And from the conclusion that PROB(freq[G|S] $\approx$ freq[G|F]) $\approx$ 1, one may *deduce* that PROB(freq[G|F] $\approx$ r) $\approx$ 1, assuming that one knows that freq(G|S) = r (where S is one's sample of observed Fs). If we suppress mention of the premise freq[{x: freq[G|x] $\approx$ freq[G|F]} | {x: $x$$\subseteq$F}] $\approx$ 1, then the preceding yields the following form of statistical induction:

(6) From S$\subseteq$F and freq[G|S] = r infer that PROB(freq[G|F] $\approx$ r) $\approx$ 1.

(6) is an instance of (1), and so we can see the manner in which statistical induction is reducible to direct inference.[7] I will now proceed upon the assumption that direct inference is relatively unproblematic, so that tracing the justificatory basis of statistical induction to direct inference yields a possible means of addressing the problem of induction. In the next section, I introduce the Goodman problem and explain why it presents, *prima facie*, a problem for the proposed approach to the problem of induction.

1 The Goodman Problem

The Goodman problem (which I will presently describe) illustrates that the unrestricted application of schemas such as (6) yields inference to mutually contradictory conclusions. Goodman's original example involves inductive inferences regarding the incidence of *green* versus *grue* emeralds (at a given time t), where an object is said to be grue *just in case* it is green and observed before t or blue and not observed before t (Goodman 1955). Under the assumption that all emeralds in our sample are green and observed before t, and any subsequently observed emeralds will be observed after t, we can construct the following instances of schema (6) (where S is our sample, E is the set of emeralds, G is the set of green objects, B is the set of blue objects, and G* is the set of grue objects, i.e., $G^* = (G \cap S) \cup (B \cap S^c)$):

(7) [Green Induction]: $S \subseteq E$ & freq[G|S] = 1.
So PROB(freq[G|E] ≈ 1) ≈ 1.

(8) [Grue Induction]: $S \subseteq E$ & freq[G*|S] = 1.
So PROB(freq[G*|E] ≈ 1) ≈1.

The problem here is that all grue emeralds that are not observed before t are blue, so that (7) and (8) yield contradictory conclusions, on the assumption that the number of emeralds that have not been observed before t is relatively large. Assume that our sample consists of 2,000 emeralds, and we know that there are at least 10,000 emeralds. In that case, the direct inferences that (may be thought to) underlie (7) and (8) are:

(9) [Green Direct Inference]: $S \in \{x\text{: } x \subseteq E \wedge |x| = 2{,}000\}$ and
$\text{freq}[\{x\text{: freq}[G|x] \approx_{0.05} \text{freq}[G|E]\} \mid \{x\text{: } x \subseteq E \wedge |x| = 2{,}000\}] \geq 0.95$.
So $\text{PROB}(\, S \in \{x\text{: freq}[G|x] \approx_{0.05} \text{freq}[G|E]\}\,) \geq 0.95$
(i.e., $\text{PROB}(\, \text{freq}[G|S] \approx_{0.05} \text{freq}[G|E]\,) \geq 0.95$).

(10) [Grue Direct Inference]: $S \in \{x\text{: } x \subseteq E \wedge |x| = 2{,}000\}$ and
$\text{freq}[\{x\text{: freq}[G^*|x] \approx_{0.05} \text{freq}[G^*|E]\} \mid \{x\text{: } x \subseteq E \wedge |x| = 2{,}000\}] \geq 0.95$.
So $\text{PROB}(\, S \in \{x\text{: freq}[G^*|x] \approx_{0.05} \text{freq}[G^*|E]\}\,) \geq 0.95$
(i.e., $\text{PROB}(\, \text{freq}[G^*|S] \approx_{0.05} \text{freq}[G^*|E]\,) \geq 0.95$).[8]

As with (7) and (8), we see that (9) and (10) yield contradictory conclusions, given the assumption that our sample is composed of 2,000 green emeralds that have been observed before t, and our assumption that there are at least 10,000 emeralds. It would seem, then, that the Goodman problem strikes at the heart of induction by direct inference. Indeed, the mathematical theorems that underlie this approach (namely, (3) and (4), above) apply equally where $\mathbb{G}$ is the set of green objects and where $\mathbb{G}$ is the set of grue objects.[9]

That instances of direct inference may lead to contradictory conclusions is not surprising. Indeed, it is well known (since Venn (1866)) that instances of direct inference may yield contradictory conclusions, which entails that instances of direct inference are *defeasible*. The problem, presented by the Goodman example, is simply that *at least one* of the two direct inferences, (9) or (10), is defeated. In the absence of some principled reason for preferring inference by (9) over inference by (10), induction by direct inference fails to address the Goodman problem, and thereby fails to be an adequate approach to the problem of induction.

In order to address the present problem, one may invoke a pre-theoretic notion of 'projectibility' as a basis for disqualifying (10), on the grounds that it is formulated using an unprojectible predicate. But in addition to being *ad hoc* (and thus unsatisfying), the move seems wrong, for there appear to be at least some cases where inductive inference using the predicate grue would be reasonable. Such cases may arise when one is wholly ignorant of the time at which the elements of one's sample were first observed (as facilitated by time travel, for example), and one knows that each element of the sample is grue, but one does not know of any element whether it is blue or whether it is green (cf. Jackson 1975). The proposal to limit induction to predicates corresponding to 'qualitative' properties, as proposed by Carnap (1947, p.146) and Swinburne (1968), also suffers from the problem that it is frequently reasonable to make inductive inferences based on observed regularities in the distribution of non-qualitative properties.[10] Rather than invoke considerations of projectibility, etc., I show that a relatively standard treatment of what some have called the "Reference Class Problem" addresses the Goodman Problem. Indeed, plausible and relatively standard principles of direct inference yield the conclusion that the Goodman inference (involving the grue predicate) is defeated, so it is unnecessary to invoke considerations of projectibility, etc., in order to address the Goodman problem.

2 The Reference Class Problem and the Defeat of the Grue Direct Inference

In making a direct inference, one assigns a probability to the statement that a given object (or event), c, is an element of a given target class, T, by locating c within a suitable reference class, R, and then concluding that the probability that c is in T is equal to r, where r is the (known) frequency of the elements of T among the elements of R. The Reference

Class Problem consists in the fact that: (i) every object is a member of numerous reference classes, and (ii) our estimates for the frequency of a given target characteristic among these different reference classes will typically vary (cf. Hájek 2007).

There are two plausible and relatively standard principles that jointly yield a *partial* solution to the Reference Class Problem:

(11) *Specificity Defeat*: If D_1 and D_2 are *admissible* direct inferences leading to mutually inconsistent conclusions (relative to one's knowledge), and the reference class for D_1 is a proper subset of the reference class for D_2, then D_2 is subject to *specificity defeat.*[11]

(12) *Rebutting Defeat*: If D_1 and D_2 are *admissible* direct inferences leading to mutually inconsistent conclusions (relative to one's knowledge), and neither the reference class for D_1 nor the reference class for D_2 is a proper subset of the other, then D_2 is subject to *rebutting defeat*, provided that D_1 is not defeated on grounds extrinsic to its conflict with D_2.[12]

Before proceeding, note that the applicability of (11) and (12) is limited to *admissible* direct inferences, where *in*admissible direct inferences are taken to be self-defeating. The inclusion of admissibility conditions within (11) and (12) is needed to handle problems associated with gerrymandered reference and target classes which arise in the context of direct inference. Absent the restriction to admissible direct inferences, (11) and (12) would yield the defeat of virtually all direct inferences, via conflict with inadmissible direct inferences (based on gerrymandered reference and target classes). The difficulty represented by inadmissible direct inferences has been variously described as a species of *projectibility* problem (Pollock 1990, Kyburg and Teng 2001) or a problem of distinguishing *relevant* from *irrelevant* statistics (Bacchus 1990, Thorn 2012). In the present article, it is assumed that the problem of determining the admissibility of direct inferences is distinct from addressing the Goodman problem. The present assumption is supported by the treatment of inadmissible direct inferences proposed by Thorn (2012): While the account of Thorn appears to correctly handle the problems associated with gerrymandered reference and target classes, the admissibility (or 'relevance') conditions introduced by Thorn do not yield a preference for [Green Direct Inference] over [Grue Direct Inference], or the conclusion that [Grue Direct Inference] is self-defeating. In keeping with the assumption that determining the admissibility of direct inferences is distinct from addressing the Goodman problem, I here proceed upon the assumption that both [Green Direct Inference] and [Grue Direct Inference] are admissible, as are the variants of [Green Direct Inference] and [Grue Direct Inference] that are introduced below.[13] In making the present assumption, I thereby foreswear the 'easy' (though ad hoc) route to defending induction by direct inference from the Goodman problem, which would consist in maintaining the admissibility of [Green Direct Inference], while denying the admissibility of [Grue Direct Inference].[14]

As formulated, (11) and (12) allow us to state the difficulty that Goodman's example presents for induction by direct inference: It appears that Specificity Defeat, (11), is inapplicable within the Goodman example, so that Rebutting Defeat, (12), applies, and yields the result that both [Green Direct Inference] and [Grue Direct Inference] are subject to rebutting defeat (since the conclusions that they generate are inconsistent, relative to one's knowledge, and neither of the relevant reference classes is a subset of the other). As it turns out, however, the two principles, (11) and (12), can be applied to dispense with [Grue Direct Inference]. In particular, it is possible to formulate a direct inference that yields the result that [Grue Direct Inference] is subject to specificity defeat. [Green Direct Inference] is, thus, not subject to rebutting defeat by [Grue Direct Inference], since [Grue Direct Inference] is defeated on grounds extrinsic to its conflict with [Green Direct Inference]. As a prelude to articulating the direct inference that yields the defeat of [Grue Direct Inference], it will be illuminating to consider a variant of Goodman's grue predicate, and an important respect in which greenness and this variant of grueness differ. The variant of the grue predicate is defined as follows: An object is m-grue (for *modified* grue) *just in case* it is green and observed before t or non-green and not observed before t. The difference to be described, between greenness and m-grueness, captures the intuition behind the proposed defeater of [Grue Direct Inference] (and the intuition behind the general result, Theorem 3, upon which the proposed defeater of [Grue Direct Inference] is based).

To allow for ease in thinking about Goodman's example, I proceed (for the moment) upon the assumption that there are 10,000 emeralds, and that our sample consists of 2,000 emeralds. Notice that the *possible* numbers of green emeralds ranges from 0 to 10,000.[15] Now consider the possible 2,000 element samples that we *could have* drawn, where the composition of the samples that we could have drawn is understood to be dependent on the total number of green emeralds. For example, if all emeralds are green, then we could not have drawn a sample that contains non-green emeralds. Call a sample "extreme-valued" with respect to greenness, if all or none of the objects in the sample are green. In the present case, we can distinguish two broad types of extreme-valued samples, i.e., ones that consist of 2,000 green emeralds and ones that consist of 2,000 non-green emeralds. Notice that within Goodman's example our sample is of the former extreme-valued type. It is now possible to show that regardless of the number of emeralds that are green among the set of all 10,000 emeralds, no more than a *very small* percentage of the (2,000 element) extreme-valued samples that we *could have* drawn *would have* agreed (approximately) with the population of all emeralds on the frequency of m-grue objects (by a margin of 5%). The proceeding holds due to the fact that an extreme-valued sample of emeralds would agree with the population of emeralds with respect to m-grueness *only if* most of the emeralds not in the sample differ from the emeralds in the sample with respect to greenness. (Note that I here assume that within each of the relevant counterfactuals, the emeralds in the sample are coextensive with the emeralds that are observed before t, within the counterfactual.) But if most of the emeralds not in our sample differ from the emeralds in our sample with respect to greenness, then very few of the (2,000 element) samples that we could have drawn would have agreed with the population of all emeralds on the frequency of m-grue objects (by a margin of 5%). We can see that the preceding claims are correct by considering the two types of possibility (concerning the number of green emeralds) where we could have drawn an extreme-valued sample that agreed (by a margin of 5%) with the population of all emeralds regarding the frequency of m-grue emeralds:[16]

(i) Suppose that the number of green emeralds is such that we could have drawn a sample composed wholly of green emeralds that agrees with the population of all emeralds with respect to the frequency of m-grueness (by a margin of 5%). Note that such a sample is composed wholly of m-grue emeralds. Moreover, if such a sample agrees with the population of all emeralds with respect to the frequency of m-grueness (by a margin of 5%), then at least 95% of all emeralds are m-grue, which implies that at least 7,500 of the 8,000 emeralds not in the sample are non-green. But in that case, the *vast majority* of the (2,000 element) extreme-valued samples that we could have drawn are composed of non-green emeralds. (In particular, the proportion of the extreme-valued samples that we could have drawn that are composed entirely of non-green emeralds is at least $\binom{7500}{2000}/(\binom{7500}{2000}+\binom{2500}{2000})$, which is greater than 0.99.[17]) Suppose we had drawn one of these samples. In that case, our sample would have been an extreme-valued sample of non-green non-m-grue emeralds, while between 5,500 and 6,000 of the 8,000 emeralds not in our sample would have been non-green m-grue emeralds. As a consequence, our sample (a representative of the vast majority of the extreme-valued samples that we could have drawn) would not have agreed with the population of all emeralds on the frequency of m-grue objects, since no emeralds in our sample are m-grue, while between 5,500 and 6,000 of the full set of 10,000 emeralds are m-grue.

(ii) Suppose that the number of green emeralds is such that we could have drawn a sample composed wholly of non-green emeralds that agrees with the population of all emeralds with respect to the frequency of m-grueness (by a margin of 5%). Note that such a sample is composed wholly of non-m-grue emeralds. Moreover, if such a sample agrees with the population of all emeralds with respect to the frequency of m-grueness (by a margin of 5%), then at least 95% of all emeralds are non-m-grue, which implies that at least 7,500 of the 8,000 emeralds not in the sample are green. But in that case, the *vast majority* of the (2,000 element) extreme-valued samples that we could have drawn are composed of green emeralds. (In particular, the proportion of the extreme-valued samples that we could have drawn that are composed entirely of green emeralds is at least $\binom{7500}{2000}/(\binom{7500}{2000}+\binom{2500}{2000})$, which is greater than 0.99.) Suppose we had drawn one of these samples. In that case, our sample would have been an extreme-valued sample of green m-grue emeralds, while between 5,500 and 6,000 of the 8,000 emeralds not in our sample would have been green non-m-grue emeralds. As a consequence our sample (a representative of the vast majority of the extreme-valued samples that we could have drawn) would not have agreed with the population of all emeralds on the frequency of m-grue objects, since all objects in the sample are m-grue, while between 5,500 and 6,000 of the full set of 10,000 emeralds are non-m-grue.

The preceding illustrates a respect in which greenness and m-grueness differ within Goodman's example: Regardless of the number of emeralds that are green among the set of all emeralds, no more than a *very small* percentage of the (2,000 element) extreme-valued samples that we *could have* drawn *would have* agreed with the population of all emeralds on the frequency of m-grue objects. In a moment, we will see just how surprisingly small this percentage is. On the other hand, there is no such limitation (i.e., no limitation that holds regardless of the total number of green emeralds) on the agreement between the samples that we could have drawn and the population of all emeralds, regarding the frequency of greenness. Indeed, if all of the 10,000 emeralds were green, then all of the samples that we could have drawn would have agreed with the population regarding greenness. In itself, the present difference between greenness and m-grueness might be sufficient to mount a relatively convincing argument for favoring induction concerning greenness over induction concerning m-grueness. That argument would be grounded in the thought that it cannot be correct to think that our extreme-valued sample of emeralds probably agrees with the population of all emeralds regarding the frequency of m-grue emeralds, since we know that very few of the (2,000 element) extreme-valued samples of emeralds that we *could have* drawn *would have* agreed with the population of all emeralds regarding the frequency of m-grue emeralds. Further, note that being grue implies being m-grue (though not vice versa). So the present reason for doubting the cogency of inductive inference concerning m-grueness also yields a reason for doubting the cogency of inductive inference concerning grueness: If it is incorrect to inductively project m-grueness (in a given situation), it must also be incorrect to inductively project grueness (in that situation), since projecting grueness entails projecting m-grueness.[18] The preceding thoughts are formulated via our intuitions about certain counterfactuals. Some might object to these intuitions. But, as it turns out, it is possible to capture the numeric features of such reasoning via a perfectly *extensional* frequency statement. In turn, this frequency statement can be used to formulate a direct inference that yields the defeat of [Grue Direct Inference], using the principles of direct inference, (11) and (12), which were proposed above. The relevant frequency statement is a theorem of mathematics (where for purposes of illustration, it is again assumed that |E| = 10,000):[19]

(13) [*mi*-Grue]: freq[{x: freq[(G∩x)∪(G^c∩x^c)|x] $\approx_{0.05}$ freq[(G∩x)∪(G^c∩x^c)|E]}
| {x: x⊆E ∧ |x|=2,000 ∧ freq[G|x]∈{0,1}}] < 8.7×10^{-100}.

[*mi*-Grue] illustrates the smallness of the percentage of extreme-valued samples that we *could have* drawn that *would have* agreed with the population of all emeralds on the frequency of *m*-grue objects. Reference to counterfactuals is avoided in expressing this value, by expedient use of the variable x in place of S (the constant which is used to denote our actual sample). Such substitutions are employed here in order to generate the 'indexicalized' set designator (G∩x)∪(G^c∩x^c), in place of the non-idexicalized designator of the set of *m*-grue of objects, i.e., (G∩S)∪(G^c∩S^c). So instead of expressing the frequency with which extreme-valued subsets of the set of emeralds agree with the set of all emeralds, regarding the frequency of *m*-grueness (as defined by (G∩S)∪(G^c∩S^c)), [*mi*-Grue] expresses the frequency with which extreme-valued subsets of the set of emeralds agree with the set of all emeralds, regarding the frequency of *mi*-grueness (for *modified indexicalized* grueness) as defined (indexically) for each respective subset, as if the respective subset *had been* S. Using [*mi*-Grue] as a major premise, it is possible to formulate the following direct inference (where S is our sample of emeralds):

(14) [*mi*-Grue Direct Inference]: S ∈ {x: x⊆E ∧ |x|=2,000 ∧ freq[G|x]∈{0,1}}
and freq[{x: freq[(G∩x)∪(G^c∩x^c)|x] $\approx_{0.05}$ freq[(G∩x)∪(G^c∩x^c)|E]}
| {x: x⊆E ∧ |x|=2,000 ∧ freq[G|x]∈{0,1}}] < 8.7×10^{-100}.
So PROB(S ∈ {x: freq[(G∩x)∪(G^c∩x^c)|S] $\approx_{0.05}$ freq[(G∩x)∪(G^c∩x^c)|E]})
< 8.7×10^{-100}
(i.e., PROB(freq[(G∩S)∪(G^c∩S^c)|S] $\approx_{0.05}$ freq[(G∩S)∪(G^c∩S^c)|E])
< 8.7×10^{-100}).

Given our knowledge that freq[(G∩S)∪(G^c∩S^c)|S] = 1, within Goodman's example, the conclusion of [*mi*-Grue Direct Inference] implies that PROB(freq[(G∩S)∪(G^c∩S^c)|E] ≥ 0.95) < 8.7×10^{-100}. But PROB(freq[(G∩S) ∪(G^c∩S^c)|E] ≥ 0.95) ≥ PROB(freq[(G∩S)∪(B∩S^c)|E] ≥ 0.95), since freq[(G∩S)∪(B∩S^c)|E] ≥ 0.95 implies that freq[(G∩S)∪(G^c∩S^c)|E] ≥ 0.95 (given (G∩S)∪(B∩S^c) ⊆ (G∩S)∪(G^c∩S^c)). Furthermore, PROB(freq[(G∩S)∪(B∩S^c) |E] ≥ 0.95) = PROB(freq[G*|E] ≥ 0.95), by definition. So [*mi*-Grue Direct Inference] supports the conclusion that PROB(freq [G*|E] ≥ 0.95) < 8.7×10^{-100}, which contradicts the conclusion of [Grue Direct Inference]. But [*mi*-Grue Direct Inference] is based on a narrower reference class than [Grue Direct Inference], i.e., $\{x\colon x\subseteq \mathrm{E} \wedge |x|= 2{,}000 \wedge \mathrm{freq}[\mathrm{G}|x]\in\{0,1\}\} \subset \{x\colon x\subseteq \mathrm{E} \wedge |x|= 2{,}000\}$. So [Grue Direct Inference] is subject to specificity defeat.[20]

In the following section, I present the theorem upon which [*mi*-Grue] is based, and discuss the limitations of this theorem as a means to addressing variations of the Goodman problem.

3 A General Result and a Discussion of Possible Limitations of the Approach

The frequency statement that was used to defeat [Grue Direct Inference] follows from the following theorem (which is proved in the Appendix):

(15) *Theorem 3*: $\forall\mathbb{E},\mathbb{G}\colon \forall n,\varepsilon>0\colon (n/|\mathbb{E}|)+\varepsilon < 1/2 \Rightarrow$
$\mathrm{freq}[\{x\colon \mathrm{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \mathrm{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\} \mid \{x\colon x\subseteq\mathbb{E} \wedge |x|=n \wedge \mathrm{freq}[\mathbb{G}|x]\in\{0,1\}\}] \leq \binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n} / \left(\binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}+\binom{|\mathbb{E}|-\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}\right)$.[21]

Theorem 3 can be used to generate variations of [*mi*-Grue], corresponding to different assumptions about the size of S (our sample), and the size of E (the set of all emeralds). Such variations of [*mi*-Grue] may be used to defeat corresponding variations of [Grue Direct Inference]. Observe, moreover, that the specified upper bound for respective instances of $\mathrm{freq}[\{x\colon \mathrm{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \mathrm{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\} \mid \{x\colon x\subseteq\mathbb{E} \wedge |x|=n \wedge \mathrm{freq}[\mathbb{G}|x]\in\{0,1\}\}]$ will be *very small* so long as $|\mathbb{E}|-\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor$ is not small, and is somewhat greater than $\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor$. This is illustrated by the example of [*mi*-Grue] (above). There are, of course, variations of Goodman's example for which Theorem 3 will not generate an appropriate defeater. In the following subsections, I sketch the means by which such variations of Goodman's example may be addressed.

3.1 The Sample Contains Some non-Green Emeralds

Note that the reference class mentioned in Theorem 3 only applies to samples that satisfy freq[$\mathbb{G}|x$]$\in$$\{0,1\}$ (for respective $\mathbb{G}$). This means that the theorem cannot be used to handle cases where the frequency of green objects among our sample is not extreme-valued (i.e., cases where freq[G|S]$\notin$$\{0,1\}$). Fortunately, results similar to the one expressed by Theorem 3 can be generated for cases where freq[G|S]$\notin$$\{0,1\}$, so long as the value of freq[G|S] differs somewhat from 0.5.[22] For example, in the case where |E|=10,000, |S|=2,000, and freq[G|S] = 0.6, it turns out (as a matter of mathematical necessity) that freq[$\{x$: freq[$(\mathrm{G}\cap x)\cup(\mathrm{G^c}\cap x^{\mathrm{c}})|x$] $\approx_{0.05}$ freq[$(\mathrm{G}\cap x)\cup(\mathrm{G^c}\cap x^{\mathrm{c}})|\mathrm{E}$]$\}$ | $\{x$: $x\subseteq\mathrm{E} \wedge |x|=2{,}000 \wedge$ freq[$\mathrm{G}|x$]$\notin(0.4,0.6)\}$] $< 7.5\times10^{-10}$. So in the case |E|=10,000, |S|=2,000, and freq[G|S] = 0.6, we may formulate a direct inference (using the proceeding frequency statement as a major premise) that entails the specificity defeat of the respective variation of [Grue Direct Inference]. On the other hand, cases where the value of freq[G|S] does not differ significantly from 0.5 are not a problem for the proposed approach to the Goodman problem, for in such cases [Grue Direct Inference] delivers conclusions that are consistent with the ones delivered by [Green Direct Inference].

3.2 The Relative Size of the Sample is Large

Now note that Theorem 3 applies according to the relative size of one's sample in comparison to the population from which the sample was drawn (as represented by the quotient $n/|\mathbb{E}|$). A clear limitation of Theorem 3, when applied to variations of the Goodman example, concerns cases where |S|/|E| is greater than or close to 1/2, in which case Theorem 3 supplies no defeater for the respective variation of [Grue Direct Inference].

While Theorem 3 is inapplicable, it is possible to address variations of the Goodman example where the relative size of one's sample in comparison to the population from which it was drawn approaches or exceeds 1/2. For example, within a Goodman type example, suppose that |S|= 2,000, and it is known that |E|= 4,000. In this case, one may simply accept that the corresponding variants of [Green Direct Inference] and [Grue Direct Inference] are subject to rebutting defeat (in the absence of an applicable variant of [mi-Grue]). As a means to drawing an appropriate conclusion about freq[G|E], one may instead base one's induction emulating direct inference upon a suitably small, and representative, subset of one's sample (say a subset s, where |s| = 1,000), and a modified variant of [Green Direct Inference], where the major premise is the mathematical truth that freq[{$\langle x, y\rangle$: freq[G|x] $\approx_{0.05}$ freq[G|E]} | {$\langle x, y\rangle$: $x \subseteq y \subseteq$E $\wedge$ |x|= 1,000 $\wedge$ |y|= 2,000 $\wedge$ freq[G|x] = freq[G|y]}] $\geq$ 0.999999999. In particular:

(17) [Green Direct Inference Variation]: $\langle$s, S$\rangle$ $\in$ {$\langle x, y\rangle$: $x \subseteq y \subseteq$E $\wedge$ |x|= 1,000 $\wedge$ |y|= 2,000 $\wedge$ freq[G|x] = freq[G|y]} and
freq[{$\langle x, y\rangle$: freq[G|x] $\approx_{0.05}$ freq[G|E]} | {$\langle x, y\rangle$: $x \subseteq y \subseteq$E $\wedge$ |x|= 1,000 $\wedge$ |y|= 2,000 $\wedge$ freq[G|x] = freq[G|y]}] $\geq$ 0.999999999.
So PROB($\langle$s, S$\rangle$ $\in$ {$\langle x, y\rangle$: freq[G|x] $\approx_{0.05}$ freq[G|E]}) $\geq$ 0.999999999
(i.e., PROB(freq[G|s] $\approx_{0.05}$ freq[G|E]) $\geq$ 0.999999999).

In this case, the parallel variant of [Grue Direct Inference] will be subject to specificity defeat via a suitable variant of [mi-Grue Direct Inference], that employs the following (necessarily true) major premise: freq[{$\langle x, y\rangle$: freq[(G$\cap x$)$\cup$(G$^{c}\cap x^{c}$)|x] $\approx_{0.05}$ freq[(G$\cap x$)$\cup$(G$^{c}\cap x^{c}$)|E]} | {$\langle x, y\rangle$: $x \subseteq y \subseteq$E $\wedge$ |x|=$1,000$ $\wedge$ |y|=$2,000$ $\wedge$ freq[G|x] = freq[G|y]$\in${0,1} } = 0. In Particular:

(18) [mi-Grue Direct Inference Variation]: $\langle$s, S$\rangle$ $\in$ {$\langle x, y\rangle$: $x \subseteq y \subseteq$E $\wedge$ |x|= 1,000 $\wedge$ |y|= 2,000 $\wedge$ freq[G|x] = freq[G|y]$\in${0,1}} and
freq[{$\langle x, y\rangle$: freq[(G$\cap x$)$\cup$(G$^{c}\cap x^{c}$)|x] $\approx_{0.05}$ freq[(G$\cap x$)$\cup$(G$^{c}\cap x^{c}$)|E]} | {$\langle x, y\rangle$: $x \subseteq y \subseteq$E $\wedge$ |x|=$1,000$ $\wedge$ |y|=$2,000$ $\wedge$ freq[G|x] = freq[G|y]$\in${0,1} } = 0.
So PROB($\langle s, S\rangle \in${$\langle x, y\rangle$: freq[(G$\cap x$)$\cup$(G$^{c}\cap x^{c}$)|x] $\approx_{0.05}$ freq[(G$\cap x$)$\cup$(G$^{c}\cap x^{c}$)|E]}) = 0
(i.e., PROB(freq[(G$\cap$s)$\cup$(G$^{c}\cap$s^{c})|s] $\approx_{0.05}$ freq[(G$\cap$s)$\cup$(G$^{c}\cap$s^{c})|E]) = 0).

3.3 ε is Large

A final problem concerns the possibility of Goodman type inferences with major premises where the value of ε (as it appears within Theorem 3) is (or is close to) 0.5. For example, assuming |S| = 2,000, and $|E| \geq$ 10,000, we have:

(19) [Some Grue Direct Inference]: $S \in \{x\text{: } x \subseteq E \wedge |x| = 2{,}000\ \}$, and
$\text{freq}[\{x\ \text{:freq}(G^*|x) \approx_{0.5} \text{freq}(G^*|E)\} \mid \{x : x \subseteq E \wedge |x| = 2{,}000\}] \geq 0.9995$.
So PROB($S \in \{x\text{: freq}[G^*|x] \approx_{0.5} \text{freq}[G^*|E]\}$) ≥ 0.9995
(i.e., PROB($\text{freq}[G^*|S] \approx_{0.5} \text{freq}[G^*|E]$) ≥ 0.9995).

Using (19), one proposes to infer only a rough similarity between S and E, with respect to the incidence of grue objects (but with high probability). The problem with such an inference is that its conclusion is (i) inconsistent with the conclusion that one would like to draw using an instance of [Green Direct Inference], and (ii) consistent with the conclusion of all variants of [*mi*-Grue Direct Inference] (since Theorem 3 does not issue a suitable premise for variants of [*mi*-Grue Direct Inference] for values of ε that are greater than or equal to 0.5). Although the present problem is quite troubling, plausible grounds can be given for thinking that [Some Grue Direct Inference] is defeated. To see why, we must consider a peculiarity that troubles all defeasible inferences.

Suppose one knows R_1, and R_1 provides a defeasible reason for believing C. On the other hand, suppose one also knows R_2, and $R_1 \wedge R_2$ provides a defeasible reason for believing ¬C. Suppose one has no further information relevant to C, so that $R_1 \wedge R_2$ represents one's total evidence bearing on C. In that case, it is clear that one should believe ¬C, rather than C. Moreover, one's reason for believing ¬C, on the basis $R_1 \wedge R_2$, defeats the connection between R_1 and C, and thereby one's reason for believing C. But now suppose, as is plausible, that R_1 provides a defeasible reason for believing all of the logical consequences of C, including $C \vee \alpha$, where α is arbitrary. In this case, it looks like we must say that one's reason for believing ¬C (on the basis $R_1 \wedge R_2$) also defeats the connection between R_1 and $C \vee \alpha$, and thereby one's reason for believing $C \vee \alpha$ (especially since $C \vee \alpha$ taken together with ¬C implies α). Following the terminology of Thorn (2014), we may say, in this case, that one's reason for believing $C \vee \alpha$ is *derivative* of one's reason for believing C, and note that a derivative reason is defeated when the reason from which it is derived is defeated (cf. Pollock 1995).

As pointed out by Thorn (2014), a problem related to the issue of derivative reasons concerns *partially derivative* reasons. To get the idea, consider the following probabilistic variation of the preceding example. Suppose one knows R_1, and R_1 provides a defeasible reason for believing/adopting $PROB(C) \geq 0.95$. In addition, suppose one knows R_2, and $R_1 \wedge R_2$ provides a defeasible reason for believing $PROB(\neg C) \geq 0.99$. Once again, it is clear that one should believe $PROB(\neg C) \geq 0.99$, rather than $PROB(C) \geq 0.95$ (assuming one has no further information bearing on C). But now suppose that R_1 also provides a defeasible reason for believing $PROB(C \vee \alpha) \geq 0.99$ (where $PROB(C \vee \alpha) \geq 0.99$ is consistent with $PROB(\neg C) \geq 0.99$), and one has no further information relevant to C, α, and $C \vee \alpha$. In that case, one's reason for believing $PROB(C \vee \alpha) \geq 0.99$ is closely related to one's reason for believing $PROB(C) \geq 0.95$, but it would be incorrect to say that one's reason for the former conclusion is (wholly) derivative of one's reason for the latter, since the former conclusion does not entail the latter. Nevertheless, it appears that one's reason for $PROB(C \vee \alpha) \geq 0.99$ is 'partly derivative' of one's reason for $PROB(C) \geq 0.95$, since they have a shared basis, R_1, and C entails $C \vee \alpha$. Moreover, the defeat of one's reason for believing $PROB(C) \geq 0.95$ impugns one's reason for believing $PROB(C \vee \alpha) \geq 0.99$, to some degree. If the following point is not clear on its face, note that taken together with $PROB(\neg C) \geq 0.99$, $PROB(C \vee \alpha) \geq 0.99$ implies $PROB(\alpha) \geq 0.98$, and it is implausible to think that in the circumstances described an agent is entitled to believe that $PROB(\alpha) \geq 0.98$.

Following Thorn (2014), it is plausible to adopt a 'skeptical policy' with respect to conclusions that are partially derivative of defeated reasons, and assume that such reasons are defeated to the degree that their content is derivative of the defeated reason. In the case just described, one's reason for $PROB(C \vee \alpha) \geq 0.99$ is partially derivative of one's reason for $PROB(C) \geq 0.95$, so it is reasonable to conclude that the correct greatest lower probability bound for $C \vee \alpha$ should be equal to one's greatest defeasible lower probability bound for $C \vee \alpha$ (i.e., 0.99) minus one's greatest defeasible lower probability bound for C (i.e., 0.95), with the result that the correct conclusion to draw (regarding $C \vee \alpha$) is that $PROB(C \vee \alpha) \geq 0.04$. In general, in cases where an otherwise undefeated defeasible reason is partially derivative of another defeated reason, I propose that the greatest lower posterior probability bound that one accepts on the basis of the partially derivative reason is $pd-d$, where pd is the greatest defeasible lower probability bound specified by the partially derivative reason, and d is the greatest defeasible lower probability bound specified by the defeated deriving reason (cf. Thorn 2014).

The preceding account of partially derivative reasons entails the defeat of [Some Grue Direct Inference], since the reason associated with that inference is partially derivative of the reason associated with [Grue Direct Inference] (which is defeated via [*mi*-Grue Direct Inference]). Given the assumed values for |E| and |S|, the grounds specified by [Some Grue Direct Inference] supply an acceptable reason for believing that PROB(freq(G*|S) $\approx_{0.5}$ freq(G*|E)) $\geq$ 0.0495 (since $0.9995 - 0.95 = 0.0495$). This conclusion (i.e., that PROB(freq(G*|S) $\approx_{0.5}$ freq(G*|E)) $\geq$ 0.0495) is consistent with the intuitively correct conclusion (derived via [Green Direct Inference]) that PROB(freq(G|S) $\approx_{0.05}$ freq(G|E)) $\geq$ 0.95. While a slightly less skeptical approach to partially derivative reasons may be tenable (such as the one proposed, in (Thorn, 2014, fn. 13)), it is doubtful that any tenable approach yields the result that the grounds specified by [Some Grue Direct Inference] supply an acceptable reason for a conclusion that will rebut [Green Direct Inference].

4 Conclusion

In the present article, I have shown that a certain approach to the problem of induction, which I call "induction by direct inference", comfortably handles Goodman's problem of induction. In particular, I demonstrated that relatively standard principles of direct inference yield the conclusion that the Goodman inference involving the grue predicate is defeated. In addition to addressing the basic version of the Goodman example (where the relative size of one's sample is small, etc.), I sketched the means of applying the approach in order to address possible variants of Goodman's example.

5 *Appendix*

Lemma 1: $\forall\mathbb{E},\mathbb{G},\mathbb{S},\varepsilon$: $\mathbb{S}\subseteq\mathbb{E} \wedge \text{freq}[\mathbb{G}|\mathbb{S}] \approx_\varepsilon \text{freq}[(\mathbb{G}\cap\mathbb{S})\cup(\mathbb{G}^c\cap\mathbb{S}^c)|\mathbb{E}] \Rightarrow$
$\text{freq}[\mathbb{G}|\mathbb{S}] \approx_{\varepsilon+(|\mathbb{S}|/|\mathbb{E}|)} \text{freq}[\mathbb{G}^c|\mathbb{E}]$.

Proof: $\text{freq}[(\mathbb{G}\cap\mathbb{S})\cup(\mathbb{G}^c\cap\mathbb{S}^c)|\mathbb{E}]$ differs from $\text{freq}[\mathbb{G}^c|\mathbb{E}]$ by at most $|\mathbb{S}|/|\mathbb{E}|$, since $\text{freq}[(\mathbb{G}\cap\mathbb{S})\cup(\mathbb{G}^c\cap\mathbb{S}^c)|\mathbb{E}] = |(\mathbb{G}\cap\mathbb{S})|/|\mathbb{E}|+|(\mathbb{G}^c\cap\mathbb{S}^c)\cap\mathbb{E}|/|\mathbb{E}|$ and
$\text{freq}[\mathbb{G}^c|\mathbb{E}] = |(\mathbb{G}^c\cap\mathbb{S})|/|\mathbb{E}|+|(\mathbb{G}^c\cap\mathbb{S}^c)\cap\mathbb{E}|/|\mathbb{E}|$.□

Lemma 2:
$\forall\mathbb{E},\mathbb{G},n,\varepsilon$: $\text{freq}[\{x\text{: freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \text{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\}$
$|\{x\text{: } x\subseteq\mathbb{E} \wedge |X|=n \wedge \text{freq}[\mathbb{G}|x]\in\{0,1\}\}] > 0 \Rightarrow$
$(\text{freq}[\mathbb{G}|\mathbb{E}] \approx_{\varepsilon+(n/|\mathbb{E}|)} 0 \;\vee\; \text{freq}[\mathbb{G}|\mathbb{E}] \approx_{\varepsilon+(n/|\mathbb{E}|)} 1)$.

Proof: Assume the negation of Lemma 2, i.e., $\exists\mathbb{E},\mathbb{G},n,\varepsilon$: $\text{freq}[\{x\text{: freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \text{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\} \mid \{x\text{: } x\subseteq\mathbb{E} \wedge |X|=n \wedge \text{freq}[\mathbb{G}|x] \in\{0,1\}\}] > 0$, and not $\text{freq}[\mathbb{G}|\mathbb{E}] \approx_{\varepsilon+(n/|\mathbb{E}|)} 0$ and not $\text{freq}[\mathbb{G}|\mathbb{E}] \approx_{\varepsilon+(n/|\mathbb{E}|)} 1$. Consider the two sorts of sets that are elements of $\{x\text{: } x\subseteq\mathbb{E} \wedge |x|=n \wedge \text{freq}[\mathbb{G}|x]\in\{0,1\}\}$, i.e., the ones that satisfy (i) $x\subseteq\mathbb{E} \wedge |x|=n \wedge \text{freq}[\mathbb{G}|x] = 0$, and the ones that satisfy (ii) $x\subseteq\mathbb{E} \wedge |x|=n \wedge \text{freq}[\mathbb{G}|x] = 1$. The sets that satisfy (i) are not elements of $\{x\text{: freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \text{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\}$. Assume there was such an x, called "S". Then $\text{freq}[\mathbb{G}|\text{S}] \approx_\varepsilon \text{freq}[(\mathbb{G}\cup\text{S})\cap(\mathbb{G}^c\cup\text{S}^c)|\mathbb{E}]$, and thus $\text{freq}[\mathbb{G}^c|\mathbb{E}] \approx_{\varepsilon+(n/|\text{E}|)} 0$ (by Lemma 1), and thus $\text{freq}[\mathbb{G}|\mathbb{E}] \approx_{\varepsilon+(n/|\text{E}|)} 1$, contrary to our initial assumptions. Similarly, the sets that satisfy (ii) are not elements of $\{x\text{: freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \text{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\}$. Assume there was such an x, called "S". Then $\text{freq}[\mathbb{G}|\text{S}] \approx_\varepsilon \text{freq}[(\mathbb{G}\cup\text{S})\cap(\mathbb{G}^c\cup\text{S}^c)|\mathbb{E}]$, and thus $\text{freq}[\mathbb{G}^c|\mathbb{E}] \approx_{\varepsilon+(n/|\mathbb{E}|)} 1$ (by Lemma 1), and thus $\text{freq}[\mathbb{G}|\mathbb{E}] \approx_{\varepsilon+(n/|\mathbb{E}|)} 0$, contrary to our initial assumptions.□

Theorem 3: $\forall\mathbb{E},\mathbb{G}$: $\forall n,\varepsilon>0$: $\varepsilon+(n/|\mathbb{E}|) < 1/2 \Rightarrow$
$\text{freq}[\{x\text{: freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \text{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\} \mid \{x\text{: } x\subseteq\mathbb{E} \wedge |x|=n \wedge \text{freq}[\mathbb{G}|x]\in\{0,1\}\}] \leq$
$\binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}/(\binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}+\binom{|\mathbb{E}|-\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}))$.

Proof: Let $\mathbb{E}$, $\mathbb{G}$, n, and $\varepsilon>0$ be arbitrary, such that $\varepsilon+(n/|\mathbb{E}|) < 1/2$. Assume that $\text{freq}[\{x\text{: freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|x] \approx_\varepsilon \text{freq}[(\mathbb{G}\cap x)\cup(\mathbb{G}^c\cap x^c)|\mathbb{E}]\} \mid \{x\text{: } x\subseteq\mathbb{E} \wedge |x|=n \wedge \text{freq}[\mathbb{G}|x]\in\{0,1\}\}] > 0$.

Then one of two cases obtains, namely: [1] freq[$\mathbb{G}|\mathbb{E}$] $\approx_{\varepsilon+(n/|\mathbb{E}|)}$ 0, or [2] freq[$\mathbb{G}|\mathbb{E}$] $\approx_{\varepsilon+(n/|\mathbb{E}|)}$ 1 (by Lemma 2).

Assume [1], and consider the two sorts of sets that are elements of $\{x\colon x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq[$\mathbb{G}|x$]$\in\{0,1\}\}$, i.e., the ones that satisfy (i) $x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq[$\mathbb{G}|x$] $= 0$, and the ones that satisfy (ii) $x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq[$\mathbb{G}|x$] $= 1$. The sets that satisfy (i) do not satisfy freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|x$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|\mathbb{E}$]. Assume there was such an x, called "S". Then freq[$\mathbb{G}|\mathrm{S}$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cup\mathrm{S})\cap(\mathbb{G}^{\mathrm{c}}\cup\mathrm{S}^{\mathrm{c}})|\mathbb{E}$], and thus freq[$\mathbb{G}|\mathrm{S}$] $\approx_{\varepsilon+(|n/|\mathbb{E}|)}$ freq[$\mathbb{G}^{\mathrm{c}}|\mathbb{E}$] (by Lemma 1). But freq[$\mathbb{G}|\mathrm{S}$] $= 0$, and so freq[$\mathbb{G}^{\mathrm{c}}|\mathbb{E}$] $\approx_{\varepsilon+(n/|\mathbb{E}|)}$ 0, which is contrary to [1] (given that $\varepsilon+(n/|\mathbb{E}|) < 1/2$). The sets that satisfy (ii) also satisfy freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|x$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|\mathbb{E}$], given [1]. (This follows for such a set S, since $|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}}\cap\mathbb{E}|+|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}\cap\mathbb{E}|= |\mathbb{G}^{\mathrm{c}}\cap\mathbb{E}|$, and $|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}\cap\mathbb{E}|= 0$, by (ii). So $|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}}\cup\mathbb{E}|= |\mathbb{G}^{\mathrm{c}}\cap\mathbb{E}|$. But $|\mathbb{G}^{\mathrm{c}}\cap\mathbb{E}|/|\mathbb{E}| \geq 1-(\varepsilon+(n/|\mathbb{E}|))$, by [1]. So $|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}}\cap\mathbb{E}|/|\mathbb{E}| \geq 1-(\varepsilon+(n/|\mathbb{E}|))$, and so $|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}} \cap\mathbb{E}| \geq |\mathbb{E}|- \varepsilon\times|\mathbb{E}|- n$. But freq[$(\mathbb{G}\cap\mathrm{S})\cup(\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}})|\mathbb{E}$] $= (|\mathbb{G}\cap\mathrm{S}|+|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}}\cap\mathbb{E}|) /|\mathbb{E}|= (n+|\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}}\cap\mathbb{E}|)/|\mathbb{E}|$, by (ii). So freq[$(\mathbb{G}\cap\mathrm{S})\cup(\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}})|\mathbb{E}$] $\geq (n + |\mathbb{E}| - \varepsilon\times|\mathbb{E}| - n)/|\mathbb{E}| = 1-\varepsilon$.) The number of such subsets of $\mathbb{E}$ is $\binom{|\mathbb{E}\cap\mathbb{G}|}{n}$, which is maximized when $|\mathbb{E}\cap\mathbb{G}| = \lfloor\varepsilon\times|\mathbb{E}| + n\rfloor$ (since $|\mathbb{E}\cap\mathbb{G}| \leq \varepsilon\times|\mathbb{E}| + n$, since $|\mathbb{E}\cap\mathbb{G}|/|\mathbb{E}| \leq \varepsilon+(n/|\mathbb{E}|)$, by [1]), in which case $|\mathbb{E}\cap\mathbb{G}^{\mathrm{c}}| = |\mathbb{E}| - \lfloor\varepsilon\times|\mathbb{E}| + n\rfloor$. So, under the assumption [1], freq[$\{x\colon$ freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|x$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cap x)\cup (\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|\mathbb{E}$]$\}$ | $\{x\colon x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq[$\mathbb{G}|x$]$\in\{0,1\}\}$] $= \binom{|\mathbb{E}\cap\mathbb{G}|}{n}/(\binom{|\mathbb{E}\cap\mathbb{G}|}{n}+\binom{|\mathbb{E}\cap\mathbb{G}^{\mathrm{c}}|}{n})$. And the maximum value of freq[$\{x\colon$ freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|x$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|\mathbb{E}$]$\}$ | $\{x\colon x\subseteq\mathbb{E}\wedge |x|=n \wedge$ freq[$\mathbb{G}|x$]$\in\{0,1\}\}$] is $\binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n} /(\binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}+\binom{\mathbb{E}-\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n})$.

Assume [2], and consider the two sorts of sets that are elements of $\{x\colon x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq[$\mathbb{G}|x$]$\in\{0,1\}\}$, i.e., the ones that satisfy (i) $x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq[$\mathbb{G}|x$] $= 0$, and the ones that satisfy (ii) $x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq[$\mathbb{G}|x$] $= 1$. The sets that satisfy (ii) do not satisfy freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|x$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|\mathbb{E}$]. Assume there was such an x, called "S". Then freq[$\mathbb{G}|\mathrm{S}$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cap\mathrm{S})\cup(\mathbb{G}^{\mathrm{c}}\cap\mathrm{S}^{\mathrm{c}})|\mathbb{E}$], and thus freq[$\mathbb{G}|\mathrm{S}$] $\approx_{\varepsilon+(n/|\mathbb{E}|)}$ freq[$\mathbb{G}^{\mathrm{c}}|\mathbb{E}$] (by Lemma 1). But freq[$\mathbb{G}|\mathrm{S}$] $= 1$, and so freq[$\mathbb{G}^{\mathrm{c}}|\mathbb{E}$] $\approx_{\varepsilon+(n/|\mathbb{E}|)}$ 1, which is contrary to [2] (given that $\varepsilon+(n/|\mathbb{E}|) < 1/2$). The sets that satisfy (i), satisfy freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|x$] $\approx_\varepsilon$ freq[$(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{c}}\cap x^{\mathrm{c}})|\mathbb{E}$] (given freq[$\mathbb{G}|\mathbb{E}$] $\approx_{\varepsilon+(n/|\mathbb{E}|)}$ 1). The number of such subsets of $\mathbb{E}$ is $\binom{|\mathbb{E}\cap\mathbb{G}^{\mathrm{c}}|}{n}$, which is maximized when $|\mathbb{E}\cap\mathbb{G}^{\mathrm{c}}| = \lfloor\varepsilon\times|\mathbb{E}|+n\rfloor$ (since $|\mathbb{E}\cap\mathbb{G}^{\mathrm{c}}| \leq \varepsilon\times|\mathbb{E}|+n$, since

$|\mathbb{E}\cap\mathbb{G}^{\mathrm{C}}|/|\mathbb{E}| \leq \varepsilon+(n/|\mathbb{E}|)$, by [2]), in which case $|\mathbb{E}\cap\mathbb{G}^{\mathrm{C}}| = |\mathbb{E}| - \lfloor\varepsilon\times|\mathbb{E}|+n\rfloor$. So, under the assumption [2], freq[$\{x$: freq$[(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{C}}\cap x^{\mathrm{C}})|x] \approx_\varepsilon$ freq$[(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{C}}\cap x^{\mathrm{C}})|\mathbb{E}]\}$ | $\{x$: $x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq$[\mathbb{G}|x]\in\{0,1\}\}$] $= \binom{|\mathbb{E}\cap\mathbb{G}^{\mathrm{C}}|}{n}/(\binom{|\mathbb{E}\cap\mathbb{G}|}{n}+\binom{|\mathbb{E}\cap\mathbb{G}^{\mathrm{C}}|}{n})$, and the maximum value of freq[$\{x$: freq$[(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{C}}\cap x^{\mathrm{C}})|x] \approx_\varepsilon$ freq$[(\mathbb{G}\cap x)\cup(\mathbb{G}^{\mathrm{C}}\cap x^{\mathrm{C}})|\mathbb{E}]\}$ | $\{x$: $x\subseteq\mathbb{E} \wedge |x|=n \wedge$ freq$[\mathbb{G}|x]\in\{0,1\}\}$] is $\binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}/(\binom{\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n}+\binom{|\mathbb{E}|-\lfloor\varepsilon\times|\mathbb{E}|+n\rfloor}{n})$.□

Acknowledgements

Work on this paper was supported by DFG Grant SCHU1566/9-1 as part of the priority program "New Frameworks of Rationality" (SPP 1516). For helpful discussions, I am thankful to Ludwig Fahrbach, Gerhard Schurz, and Ioannis Votsis. Finally, I am especially thankful two anonymous referees for Kriterion who provided comments and suggestions concerning an earlier draft of the paper.

Notes

1 It is widely held that belief in such conclusions is never justified, in which case no explanation of why they are justified is to be expected.

2 Statistical induction, as exemplified by the following schema, generalizes enumerative induction. The generalization permits that r take values other than 1, and yields an explicitly probabilistic conclusion.

3 For the sake of concision, I here omit mention of (i) S's size, and (ii) the manner in which the elements of S came to be elements of one's sample, although both of these factors clearly have a bearing on the quality of inferences of the present form.

4 While many, including Venn (1866), Reichenbach (1949), Kyburg (1974), and Kyburg and Teng (2001), have assumed that the major premises for direct inference are statements of frequency or limiting frequency, other proposals have been made, in (Pollock 1990), (Bacchus 1990), and (Thorn 2012).

5 The present theorems are from (Pollock 1990, p. 71) and (McGrew 2001).

6 Note that (i) where X is a set, |X| denotes the size of X, and (ii) where u, v, and w are numbers $u \approx_w v$ expresses that u and v differ by no more than w.

7 It is here acknowledged that a respective instance of (5) will frequently be superceded (and thereby defeated) by a direct inference based on a frequency statement whose reference class only includes sets whose size is similar (or identical) to the size of S. The result, in such cases, so long as |S| is large, is a corresponding variation of (6) which locates S among the reference class consisting of subsets of F whose size is similar (or identical) to the size of S (cf. fn. 3).

8 The values for the statistical premises of (9) and (10) are inferable via (4).

9 Similar observations have been made by Lange (2008), and John T. Roberts (in his presentation at the 2012 Annual Conference of the British Society for the Philosophy of Science).

10 A similar objection may be raised against the proposal of Quine (1969) to limit induction to predicates that correspond to 'natural kinds'.

11 Proposals such as the one codified by (11) are found in the accounts of direct inference proposed by Venn (1866), Reichenbach (1949), Pollock (1990), Bacchus (1990), Kyburg and Teng (2001), and Thorn (2012).

12 Proposals such as the one codified by (12) are found in the accounts of direct inference proposed by Reichenbach (1949), Pollock (1990), Kyburg and Teng (2001), and Thorn (2012).

13 All of these direct inferences are admissible according to the account of direct inference proposed by Thorn (2012).

14 As with the treatment of the Goodman problem proposed here, the account of direct inference proposed in (Thorn 2012) does not invoke considerations of projectibility, in determining which direct inferences are admissible.

15 Alternatively, we may suppose that there are at least 2,000 green emeralds. This will not make any difference to the reasoning presented here.

16 If neither of these possibilities obtain, then we could not have drawn an extreme-valued sample that agrees with the population with respect to m-grueness.

17 Note that $\binom{n}{k}$ denotes the *binomial coefficient* of n and k, so that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $n \geq k \geq 0$.

18 Similar reasoning undermines the cogency of induction regarding non-grueness, in the case where our sample consists of non-green emeralds.

19 The general theorem upon which [mi-Grue] is based is introduced in the following section.

20 Under the conditions outlined within Goodman's example, there is no statement, paralleling [mi-Grue], (that one is in a position to know) that expresses that samples that are extreme-valued with respect to m-grueness (or grueness) frequently disagree with E on the frequency of greenness. For example, if all emeralds are green, then our sample is the only 2,000 member set of emeralds that is extreme-valued with respect to m-grueness (and grueness), and our sample agrees with the set of all emeralds on the frequency of greenness.

21 $\lfloor r \rfloor$ denotes the result of rounding r down to the nearest natural number.

22 As with the generalizations mentioned below, a precise expression of the present result would be unwieldy. For this reason, I rely on examples to illustrate these results. The values presented within these examples were computed using a computer program written in VB.NET 2010.

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