KRITERION, Nr.4 (1992), pp.20–24

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COMPLEMENTARY LOGICS FOR CLASSICAL PROPOSITIONAL LANGUAGES

This note presents a simple axiomatic system by means of which exactly those sentences can be derived that are rated non-tautologous in classical propositional logic. Since the logic is decidable, there exist of course many algorithms that do the job, e.g. using semantic tableaux or refutation trees. However, a formulation in terms of axioms and rules of inference is by no means a straightforward task, as these must be of a most non-standard non-classical sort.[1] For instance, axioms cannot be axiom schemata and standard substitution rules cannot hold, since a non-tautology may well become tautologous upon substitution. Moreover, the system must be paraconsistent, i.e. such as to allow derivation of sentences with opposite truth values. The system presented here provides, I think, a rather nice way of dealing with these difficulties.

Since tautologous sentences are also axiomatizable, the outcome is an exhaustive characterization of the logic of classical propositional languages *in purely syntactic terms*. The picture can then be completed by developing related systems axiomatizing classical contradictions, contingencies, non-contradictions and non-contingencies, respectively: systems of this kind, which provide additional examples of paraconsistent calculi with a classical background, are discussed in the final section.

1. Preliminaries

I shall focus on a propositional language $\mathcal{L}$ the symbols of which comprise a denumerable stock of *variables* 'p_0', 'p_1', 'p_2', ..., the three connectives '~' (negation), '∧' (conjunction), '∨' (disjunction), and the two parentheses '(' and ')'[2]. The notion of a *sentence* of $\mathcal{L}$ is defined as usual: every variable is a sentence; and if A, B and C are sentences, so are $\sim A$, $(B \wedge C)$ and $(B \vee C)$. Variables are also called *atomic* sentences. In addition, sentences involving exactly the same variables will be referred to as *conjoint* sentences, whereas sentences with no variable in common will qualify as *disjoint*. I shall also speak of two sentences A and B as being *equivalent* (to each other): by this I shall mean that one of the following conditions holds for some sentences $C, C_0, C_1, C_2, \ldots, C_n$, where $n \geq 2$:

E.1 $A, B \in \{C, \sim\sim C\}$
E.2 $A, B \in \{C, (C \wedge C)\}$
$\quad A, B \in \{C, (C \vee C)\}$
E.3 $A, B \in \{(C_1 \wedge C_2), (C_2 \wedge C_1)\}$
$\quad A, B \in \{(C_1 \vee C_2), (C_2 \vee C_1)\}$
E.4 $A, B \in \{(\sim C_1 \wedge \sim C_2), \sim(C_2 \vee C_1)\}$
$\quad A, B \in \{(\sim C_1 \vee \sim C_2), \sim(C_2 \wedge C_1)\}$
E.5 $A, B \in \{(C \wedge (\sim C_2 \vee C_2 \vee \ldots \vee C_n)), C\}$
$\quad A, B \in \{(C \vee (\sim C_2 \wedge C_2 \wedge \ldots \wedge C_n)), C\}$
E.6 $A, B \in \{(C_0 \wedge (C_1 \wedge C_2)), ((C_0 \wedge C_1) \wedge C_2)\}$
$\quad A, B \in \{(C_0 \vee (C_1 \vee C_2)), ((C_0 \vee C_1) \vee C_2)\}$
E.7 $A, B \in \{(C_0 \wedge (C_1 \vee C_2)), ((C_0 \wedge C_1) \vee (C_0 \wedge C_2))\}$
$\quad A, B \in \{(C_0 \vee (C_1 \wedge C_2)), ((C_0 \vee C_1) \wedge (C_0 \vee C_2))\}$

Here '$(\sim C_2 \vee C_2 \vee \ldots \vee C_n)$' and '$(\sim C_2 \wedge C_2 \wedge \ldots \wedge C_n)$' are meant to stand for any sentence of $\mathcal{L}$ that can be obtained from such expressions by replacing each C_i by a sentence ($2 \leq i \leq n$) and a judicious addition of parentheses. In the following, I shall take for granted this and other obvious abbreviations unless the context would not prevent confusion.

2. Semantics and syntactics

Relative to the language $\mathcal{L}$, we may define a *valuation* to be any function $\mathcal{V}$ assigning a value in {0, 1} to each sentence, subject to the usual conditions:

V.1 if $A = \sim B$, then $\mathcal{V}(A) = 1 - \mathcal{V}(B)$
V.2 if $A = (B \wedge C)$, then $\mathcal{V}(A) = \mathcal{V}(B) \cap \mathcal{V}(C)$
V.3 if $A = (B \vee C)$, then $\mathcal{V}(A) = \mathcal{V}(B) \cup \mathcal{V}(C)$
V.4 if A is obtained from a sentence A' by replacing one or more occurrences of a sentence B' in A' by occurrences of a sentence B to which B' is equivalent, then $\mathcal{V}(A) = \mathcal{V}(A')$[3]

1. A complete, Hilbert-type axiomatization of classical non-tautologies was first provided by Caicedo (1978). Following Łukasiewicz's (1951) rejection method, in Varzi (1990) I have introduced a simplified system based on the sole axiom ⊥ (falsehood) and the two rules of inference: (i) if A is a substitution instance of B, then $A \vdash B$; and (ii) if A is obtained from B by replacement of equivalent sentences, then $B \vdash A$ (counting as equivalent the pairs {⊤, ~⊥}, {⊥, ⊥ ∨ ⊥}, {⊤, ⊥ ∨ ⊤}, {⊤, ⊤ ∨ ⊥}, {⊤, ⊤ ∨ ⊤}). Similar methods have been used independently by Scott (1957) and Dutkiewicz (1989) to axiomatize intuitionistic non-tautologies. The system presented in this note is mathematically less elegant, but the format of the inferential rules and the structure of the completeness proof possess some intrinsic interest and allow useful comparisons with the logic of tautologies.

2. The choice of connectives is dictated merely by expository convenience. Any truth-functionally complete set of connectives (such as '~' and '∧', or '~' and '∨') would do.
3. V.4 is obviously implied by V.1–V.3. I only include it here for convenient reference.

On this basis, and taking 0 as the antidesignated value, a given sentence A of $\mathcal{L}$ will then qualify as a *non-tautology* if and only if (iff) there exists a valuation $\mathcal{V}$ such that $\mathcal{V}(A) = 0$ (in the following, I shall refer to such a $\mathcal{V}$ as a *non-model* of A).

To provide a syntactic counterpart of this notion, let every variable of $\mathcal{L}$ count as an *axiom*, and let a sentence A count as a *consequence* of a set of sentences Γ iff one of the following conditions is satisfied:

C.1 $A = {\sim}B$ for some atomic sentence $B \in \Gamma$
C.2 $A = (B \wedge C)$ for some conjoint sentences $B, C \in \Gamma$
C.3 $A = (B \vee C)$ for some disjoint sentences $B, C \in \Gamma$
C.4 A is obtained from a sentence $A' \in \Gamma$ by replacing one or more occurrences of a sentence B' in A' by occurrences of a sentence B to which B' is equivalent[4]

Let now a sentence A count as a *non-theorem* (of $\mathcal{L}$) iff there exists a sequence of sentences $A_0 \dots A_n$ ($n \geq 0$) so that (i) $A_n = A$, and (ii) each A_i ($i \leq n$) is either an axiom or a consequence of $\{A_0 \dots A_{i-1}\}$ (such a sequence may be called a *non-proof* of A). Then, as we shall now see, the notion of a non-tautology and the notion of a non-theorem identify exactly the same sentences.

3. *Soundness and completeness*

That the set of all non-tautologies includes the set of all non-theorems is easily seen, i.e.:

T.1 *Every non-theorem is a non-tautology.*

For, suppose that A is a non-theorem, and let $A_0 \dots A_n$ ($n \geq 0$) be a non-proof of $A = A_n$. Clearly, A_0 is non-tautologous, since A_0 can only be a variable and every variable has infinitely many non-models. So pick any other A_k ($0 < k \leq n$) and assume that each A_i ($0 \leq i < k$) has a non-model. Let $\Gamma = \{A_0, \dots, A_{k-1}\}$. If $A_k = {\sim}B$ for some atomic sentence $B \in \Gamma$ (by C.1), then again B has to be a variable, and so any of the infinitely many valuations that assign 1 to B will be a non-model of A_k (by V.1). Next, if $A_k = (B \wedge C)$ for some conjoint sentences $B, C \in \Gamma$ (by C.2), then any non-model $\mathcal{V}$ of B will surely be a non-model of A_k, for $\mathcal{V}(B) = 0$ implies $\mathcal{V}(B \wedge C) = \mathcal{V}(A_k) = 0$ (by V.2). Third, if $A_k = (B \vee C)$ for some disjoint sentences $B, C \in \Gamma$ (by C.3), then a non-model of A_k can be constructed as follows: let $\mathcal{V}_1$ be a non-model of B, let $\mathcal{V}_2$ be a non-model of C, and let $\mathcal{V}$ be that valuation which agrees with $\mathcal{V}_1$ with respect to the value to be assigned to the variables occurring in B, and agrees with $\mathcal{V}_2$ with respect to the value to be assigned to any other variable (including those occurring in C); then clearly $\mathcal{V}(B) = \mathcal{V}_1(B)$ and $\mathcal{V}(C) = \mathcal{V}_2(C)$, hence $\mathcal{V}(B \vee C) = \mathcal{V}(A_k) = 0$ (by V.3). Finally, if A_k is obtained from some sentence $A' \in \Gamma$ by replacing one or more occurrences of a sentence B' in A' by occurrences of a sentence B to which B' is equivalent (by C.4), then any non-model of A' will also be a non-model of A_k, as every valuation that assigns the value 0 to A' will assign the same value to A_k (by V.4). Thus, our arbitrary sentence A_k is sure to have a non-model. The desired result follows now by mathematical induction and generalization.

We may now proceed to prove the converse of T.1:

T.2 *Every non-tautology is a non-theorem.*

To this end, a couple of subsidiary definitions will be useful. First, let us extend the relation of equivalence introduced in Section 1 to a more general, transitive relation: I shall say that a sentence A is *indirectly equivalent* to a sentence A' (in short $A \cong A'$) iff there exists a sequence of sentences $A_0 \dots A_n$ ($n \geq 0$) such that (i) $A_n = A$, (ii) $A_0 = A'$, and (iii) A_{i+1} is equivalent to A_i for each $i < n$ (I shall call such a sequence an $\cong$-*transformation* of A' into A). Clearly, $\cong$ is an equivalence relation, and it is easy to see that if a sentence A is obtained from a sentence A' by replacing one or more occurrences of a sentence B' in A' by occurrences of a sentence B to which B' is indirectly equivalent, then $A \cong A'$. Also, an easy inductive argument (using V.4) shows that any two indirectly equivalent sentences are always assigned identical values by any valuation, which implies that $\cong$ preserves the property of being non-tautologous. Similarly, one verifies that $\cong$ preserves the property of being a non-theorem: for if $A_0 \dots A_m$ ($m \geq 0$) is a non-proof of A_m and $A_m \dots A_n$ ($n \geq m$) is an $\cong$-transformation of A_m into A_n, then $n-m$ applications of C.4 show that $A_0 \dots A_m \dots A_n$ is a non-proof of A_n. (In the following, I shall appeal to such basic properties of $\cong$ without explicit mention).

Second, let us say that a sentence A is in *standard form* (SF) iff (i) each occurence of '~' in A is immediately followed by an atomic sentence; (ii) no occurrence of '∨' in A is adjacent to a sentence that contains occurrences of '∧'; and (iii) for every conjunct C of A (= every sentence C that occurs adjacently to some occurrence of '∧' in A), all disjuncts of C (= all sen-

4. In view of C.4, it will be clear that C.1–C.3 involve several redundancies. (For example, one could add the requirements that all occurrences of '(' in A be adjacent, and that the variables occurring in A occur in a certain order; also, C.2 could be simplified to "$A = (B \wedge C)$ for some sentences B, C such that $B \in \Gamma$" and C.3 could be reduced to "$A = (B \vee C)$ or $A = (B \vee {\sim}C)$ for some sentences B, C such that $B \in \Gamma$ and C is a variable not occurring in B", in which case E.5 could then be deleted). Again, the present formulation is dictated mainly by reasons of convenience.

tences that occur adjacently to some occurrence of '$\vee$' in C) are pairwise disjoint (if A involves no conjunct, we take this condition to hold for $C = A$). Note of course that a sentence satisfying condition (i) is known in the literature as a sentence in *elementary form* (EF), whereas a sentence satisfying both (i) and (ii) is typically referred to as a sentence in (conjunctive) *normal form* (NF).

Using these auxiliary notions, I shall now prove two related lemmas, L.1 and L.2: from these, the desired result, T.2, will follow immediately.[5]

L.1 *Every non-tautology is indirectly equivalent to a sentence in standard form.*

I will break the argument into three parts: first (*a*) I will show that any given sentence A is indirectly equivalent to a sentence A' in elementary form; next (*b*) I will show that any sentence A' in elementary form is indirectly equivalent to a sentence A'' in normal form; and finally (*c*) I will show that any non-tautologous sentence A'' in normal form is indirectly equivalent to a sentence A''' in standard form. Since $\cong$ is transitive and preserves the property of being non-tautologous, L.1 will then follow from (*a*)–(*c*) by taking A to be an arbitrary non-tautology.

(*a*) The first part of the argument proceeds by mathematical induction on the length of A. Clearly, if A is atomic, then $A' = A$ is the desired sentence in EF; and if A is of the form $(B \wedge C)$ or of the form $(B \vee C)$, then we can simply put $A' = (B' \wedge C')$ or $A' = (B' \vee C')$ respectively, where B' and C' are sentences in EF such that $B \cong B'$ and $C \cong C'$ (such sentences are assumed to exist by the induction hypothesis, I.H.). If, on the other hand, A is of the form $\sim B$, then we may distinguish four cases: (i) if B is atomic, then again A is already in EF, and we may put $A' = A$; (ii) if B is of the form $\sim C$, then there must exist some sentence C' in EF such that $C \cong C'$ (by I.H.), and we may therefore put $A' = C'$ (since $A = \sim\sim C \cong C$ by E.1); (iii) if B is of the form $(C \wedge D)$, then we know that there are sentences X' and Y' in EF such that $\sim C \cong X'$ and $\sim D \cong Y'$ (by I.H.), and so we may just put $A' = (X' \vee Y')$ (since $A = \sim(C \wedge D) \cong (\sim C \vee \sim D)$ by E.4); finally (iv) if B is of the form $(C \vee D)$, then again there must be sentences X' and Y' in EF such that $\sim C \cong X'$ and $\sim D \cong Y'$ (by I.H.) and we may put $A' = (X' \wedge Y')$ (since $A = \sim(C \vee D) \cong (\sim C \wedge \sim D)$ by E.4). Thus, in every possible case, A can surely be $\cong$-transformed into a sentence A' in EF.

(*b*) Suppose now that A' is in EF. To show that A' is indirectly equivalent to some sentence A'' in NF we may, again, proceed by strong induction. The case where A' is atomic or of the form $\sim B'$ (where B' must be atomic) presents no problem, for in that case we can simply put $A'' = A'$; and if A' is a sentence of the form $(B' \wedge C')$, then B' and C' must be in EF, and so we may take any two sentences B'' and C'' in NF such that $B' \cong B''$ and $C' \cong C''$ (which are known to exist by I.H.) and set $A'' = (B'' \wedge C'')$. In case A' is of the form $(B' \vee C')$, then again there must be sentences B'' and C'' in NF such that $B' \cong B''$ and $C' \cong C''$ (by I.H.): we can then proceed by induction on the number n of occurrences of '$\wedge$' in $(B'' \vee C'')$ to show that there is a sentence Z in NF to which $(B'' \vee C'')$ is indirectly equivalent, and then set $A'' = Z$: (i) if $n = 0$, then $(B'' \vee C'')$ is already in NF, and we may therefore take $Z = (B'' \vee C'')$; if $n > 0$ and '$\wedge$' occurs in C'', then C'' must be of the form $(D_1 \wedge D_2)$, where D_1 and D_2 are sentences in NF; thus $(B'' \vee C'') = (B'' \vee (D_1 \wedge D_2)) \cong ((B'' \vee D_1) \wedge (B'' \vee D_2))$ (by E.7); but both $(B'' \vee D_1)$ and $(B'' \vee D_2)$ are assumed to be indirectly equivalent to appropriate sentences in NF (by I.H.): hence, where Z_1 and Z_2 are any two such sentences, the sentence $Z = (Z_1 \wedge Z_2)$ will do; finally (iii) if $n > 0$ but '$\wedge$' does not occur in C'', then we note that $(B'' \vee C'') \cong (C'' \vee B'')$ (by E.3), and the problem reduces to (ii).

(*c*) To conclude, assume now that A'' is in NF, and suppose A'' is a non-tautology. Then A'' must be of the form B_0 or of the form $(B_0 \wedge \ldots \wedge B_n)$ $(n > 0)$, where each B_i $(i \leq n)$ is a sentence in NF that contains no occurrence of '$\wedge$', and where at least one B_i $(i \leq n)$ involves no occurrence of opposite disjuncts (= pairs of disjuncts D_1, D_2 such that $D_1 = \sim D_2$): otherwise every valuation would assign the value 1 to A'' (by V.3). In addition, every B_j $(j \leq n)$ which does involve occurrences of opposite disjuncts can be eliminated by repeated application of E.5 (along with E.3, E.6). That is, A'' can be $\cong$-transformed into a sentence A° of the form C_0 or of the form $(C_0 \wedge \ldots \wedge C_m)$ $(m \leq n)$ where the C_k's $(k \leq m)$ are exactly those sentences B_i $(i \leq n)$ which do not involve occurrences of opposite disjuncts. Obviously, such C_k's may still involve more than one occurrence of the same variable. But such repetitions can easily be eliminated by subsequent applications of E.2 (using also E.3 and E.6). Thus, A° is in turn seen to be equivalent to a sentence A''' of the form C'_0 or of the form $(C'_0 \wedge \ldots \wedge C'_m)$ where the disjuncts of each C'_k $(k \leq m)$ are pairwise disjoint. And such a sentence, A''', is the desired sentence in SF to which A'' is indirectly equivalent.

5. If we interpret '$\cong$' as the relation of *semantic equivalence* (taking "$A \cong B$" to mean "$\mathcal{V}(A) = \mathcal{V}(B)$ for every valuation $\mathcal{V}$") or as the classical relation of *deductive equivalence* (taking "$A \cong B$" to mean "A and B can be derived from each other in the classical sentential calculus"), then L.1 and L.2 state some well-known results (Post (1921)). As they stand, the two lemmas in question are to make sure that such results only depend on V.4 and C.4.

L.2 *Every sentence in standard form is indirectly equivalent to a non-theorem.*

To see this, we can fix upon an arbitrary sentence A in SF and distinguish two cases.

(*a*) Suppose first that '$\wedge$' does not occur in A. Then A must be of the form B_0 or of the form $(B_0 \vee \ldots \vee B_n)$ $(n > 0)$, where each B_i $(i \leq n)$ is either a variable or a sentence consisting of '$\sim$' followed by a variable. Clearly, such B_i's are all non-theorems, for every variable is an axiom, and the result of prefacing '$\sim$' to an axiom always yields a non-theorem (by C.1). Also, such B_i's are all pairwise disjoint, for A is assumed to be in SF. Thus, if $A = B_0$, then A is a non-theorem, while if $A = (B_0 \vee \ldots \vee B_n)$, then A is a consequence of $\{B_0,\ldots,B_n\}$ (by n applications of C.3) whence again it follows that A is a non-theorem. And obviously, $A \cong A$.

(*b*) Suppose now that A contains occurrences of '$\wedge$'. Then A must be of the form $(B_0 \wedge \ldots \wedge B_n)$ $(n > 0)$, where each B_i $(i \leq n)$ is a sentence in NF containing no occurrence of '$\wedge$'. So let p_j $(j \geq 0)$ be any variable occurring in A: by repeated application of E.5 and E.7, A can easily be $\cong$-transformed into a sentence A' of the form $(B_0' \wedge \ldots \wedge B_n')$, where for each $i \leq n$, $B_i' = B_i$ if p_j occurs in B_i, and $B_i' = ((B_i \vee p_j) \wedge (B_i \vee \sim p_j))$ otherwise. Hence, by carrying on a finite number of such $\cong$-transformations (one for each variable occurring in A), A can eventually be $\cong$-transformed into a sentence $A°$ of the form $(C_0 \wedge \ldots \wedge C_k)$ in which '$\wedge$' occurs $k \geq n$ times and in which all the C_i's $(i \leq k)$ are pairwise conjoint. But such a sentence is a non-theorem, since it is a consequence of $\{C_0,\ldots, C_k\}$ (by k applications of C.2) where each C_i $(i \leq k)$ is a non-theorem (as shown in (*a*) above). Thus, $A°$ is the desired non-theorem to which A is indirectly equivalent.

We have thus shown that every non-tautology is indirectly equivalent to some sentence in SF (L.1) and that every sentence in SF is indirectly equivalent to some non-theorem (L.2). From this, and from the above-mentioned properties of $\cong$, it follows that every non-tautology is a non-theorem (T.2).

4. *Complementary systems*

We have seen that the syntactic system defined in Section 2, call it $\mathbf{T}^{\sim}$, is adequate to specify the set of all *non-tautologies* (= sentences that are assigned the value 0 by at least one valuation) of $\mathcal{L}$. Since the value assigned by a valuation $\mathcal{V}$ to a sentence A is always opposite to the value assigned by $\mathcal{V}$ to $\sim A$, it is clear that the set of all *non-contradictions* (= sentences that are assigned the value 1 by at least one valuation) is also specifiable by means of a purely syntactic system: just substitute '$\vee$' for '$\wedge$' and '$\wedge$' for '$\vee$' in C.2-C.3, and the resulting system, call it $\mathbf{C}^{\sim}$, will do (or: just characterize $\mathbf{C}^{\sim}$ indirectly, by defining a sentence A to be a non-theorem of $\mathbf{C}^{\sim}$ iff $\sim A$ is a non-theorem of $\mathbf{T}^{\sim}$). For the same reason, it is a fact that whenever a syntactic system $\mathbf{T}$ is given by means of which one can adequately specify the set of all *tautologies* (= sentences which are assigned the value 1 by every valuation), one can immediately define a perfectly symmetric system, $\mathbf{C}$, which is adequate to specify the set of all *contradictions* (= sentences which are assigned the value 0 by every valuation): just substitute '$\vee$' for '$\wedge$' and '$\wedge$' for '$\vee$' in the axioms and rules of inference of $\mathbf{T}$ (or: just characterize $\mathbf{C}$ indirectly, by defining a sentence A to be a theorem of $\mathbf{C}$ iff $\sim A$ is a theorem of $\mathbf{T}$).[6] Accordingly, one would get a complete picture of the logic of $\mathcal{L}$ if one could define a third pair of complementary systems, say $\mathbf{N}$ and $\mathbf{N}^{\sim}$, which are adequate to specify the set of all *contingencies* (= sentences that are both non-tautologous and non-contradictory) and the set of all *non-contingencies* (= sentences that are either tautologous or contradictory) respectively. Indirectly, such systems can of course be characterized in terms of $\mathbf{T}$, $\mathbf{C}$, $\mathbf{T}^{\sim}$ and $\mathbf{C}^{\sim}$ (just define a sentence A to be a non-theorem of $\mathbf{N}$ iff A is a non-theorem of both $\mathbf{T}^{\sim}$ and $\mathbf{C}^{\sim}$, and define A to be a theorem of $\mathbf{N}^{\sim}$ iff A is a theorem of either $\mathbf{T}$ or $\mathbf{C}$). However, the arguments of Section 3 suggest a more direct approach too. For let a sentence A count as a *Boolean* sentence iff (i) A is in standard form, (ii) all conjuncts of A are pairwise distinct, and (iii) every variable occurring in A occurs exactly once in each conjunct of A (or in A itself if A has no conjunct). Then, as axioms of $\mathbf{N}$ we can take all and only those sentences A, $\sim A$ satisfying the following condition:

A.1 A is a Boolean sentence with less than 2^n conjuncts, where n is the number of distinct variables occurring in A

and as axioms of $\mathbf{N}^{\sim}$ we can take all and only those sentences A, $\sim A$ satisfying the condition:

A.2 A is a Boolean sentence with exactly 2^n conjuncts, where n is the number of distinct variables occurring in A.

Taking a sentence A to count as a consequence of a set of sentences Γ in $\mathbf{N}$ iff C.4 holds (likewise for $\mathbf{N}^{\sim}$) and defining the notion of a non-theorem of $\mathbf{N}$ as in Section 2 (likewise for $\mathbf{N}^{\sim}$), minor changes to the proofs of T.1 and T.2 above will show that the resulting systems satisfy the desired requirements [7].

6. See for example Morgan (1973).

7. As they are, the axioms of $\mathbf{N}$ and $\mathbf{N}^{\sim}$ involve various redundancies: these can be eliminated by adding the follow-

All of these results reflect of course the fact that the relevant sets of sentences (tautologies, contradictions, contingencies, etc.) are all decidable, and should therefore come as no surprise. The point is rather that such sets are now seen to be *on a par* as far as their syntactical characterization is concerned. At the same time, this does not mean that those sets are equally "interesting". The fact remains that every substitution instance of a tautology of $\mathcal{L}$ is a valid sentence of, say, the language of quantification theory; whereas the substitution instances of a non-tautologous sentence of $\mathcal{L}$ do not have to be invalid in a language with quantifiers. They have to be non-tautologous — but that's all.[8]

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ing clauses to the definition of "Boolean sentence": (iv) all occurrences of '(' in A are adjacent, and (v) all variables occurring in A occur in alphabetical order in each conjunct of A (or in A itself if A has no conjunct). Moreover, certain axioms could be obtained as (non-)theorems if certain rules were assumed in addition to C.4. For example, it is easy to see that **N** is nothing but the weakening of $\mathbf{T}^-$ obtained by inserting A.1 in the formulation of C.2: in this sense, **N** would have the same axioms as $\mathbf{T}^-$ and a similar (though weaker) system of rules.

8. I would like to thank Bill Seager, whose work on "nether" logic (1986) was the source of my interest in the topic of this note. Bill's generous comments on an earlier draft are also gratefully acknowledged.