A Conventional Foundation of Logic[*]

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Abstract

Frege was the first one to provide a comprehensive attempt of constructing mathematics out of a logical foundation alone. In subsequent investigations also the foundations of logic were discussed quite extensively by providing fundamental principles for distinguishing logical truths from non-logical ones. Three main approaches can be differentiated in these investigations: Belnap's structural rules approach (1962), Quine's approach of substitution salva congruitate (1979), and Tarski's invariance approach (1986). All three suggestions face some problems in distinguishing adequately the logical from the non-logical vocabulary. In this paper the approach of Belnap is put a step further by providing a foundation that is conventional only.

1 Introduction

[18] It was in 1884 when Gottlob Frege tried to give a comprehensive reduction of mathematics to logic. Although Frege's logicistic programme failed in its details, most meta-mathematicians agree that it was still a very insightfull approach.

But Frege did not only try to reduce mathematics to logic, he was also one of the main figures in founding modern logic. Although carried out in a syntactic way, his motivation for accepting principles as logical principles was mainly semantic: According to him principles or laws of logic are the laws of thinking (normatively seen) and truth. It was more than a half century later when Alfred Tarski formulated a logical semantic that allowed for a precise explication of Frege's "laws of thinking and truth".

Tarski's theory presupposes a distinction of the vocabulary of an artificial language into a logical and a non-logical or descriptive one. Since the whole

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semantical enterprise hinges on such a distinction, further questions about adequate criteria for such a distinction arose. Subsequently Nuel Belnap, Tarski, and Willard van Orman Quine made proposals for such criteria—cf. (Belnap 1962), (Tarski 1986), and (Quine 1986).

We are going to concentrate here on Belnap's approach: Belnap suggests to regard as logical all those symbols, that can be introduced non-creatively or conservatively by introduction- and elimination rules into a basic system of inference or argumentation. The idea behind this criterion is that starting from a basic system of inference and argumentation, every symbol whose usage can be "explained" by help of rules of such a system must be also inferentially and argumentatively relevant, i.e. logical.

The basic system Belnap starts with is just the set of structural rules of classical inference, i.e. reflexivtiy, contraction, weakening (monotonicity), permutation, and transitivity. Since the classical logical vocabulary can be introduced and eliminated on basis of such a system, it also satisfies this criterion.

The question remains why one should start with such a basic system and not another one (e.g. with a non-monotonic one). And furthermore: What justification do we have for such a choice? In this paper we are going to provide a very pragmatic answer: It's all about conventions. All systems that can be described in accordance with basic rules for conventions are justified since conventions are more or less harmless agreements within the scientific community.

To make this point clear, the structure of our argumentation will be as follows: In the following section (2) we present in a nutshell the conventional framework we want to use. In section 3 we provide a conventional reconstruction of Belnap's basic system. Finally we draw some concluding remarks in 4.

2 Rules for Language Conventions

The perhaps strongest language conventions we use are so-called explicit definitions. By help of explicit definitions we can introduce new vocabulary into a theory in such a way, that every statement using the new vocabulary could be also equivalently formulated as statement using only the old one. Two criteria are necessary and sufficient for this, the so-called criterion of eliminability and that of non-creativity or conservativity, where $\mathcal{L}_T$ is the language $\mathcal{L}$ of theory T; $\mathcal{L}_{T,s}$ is the by the symbol s expanded language of $\mathcal{L}_T$: [19]

Criterion 1 (Eliminability) *s is eliminable in T' w.r.t T iff for all $\varphi \in \mathcal{L}_{T,s}$ there is a $\psi \in \mathcal{L}_T$ such that: $\vdash^{T'} (\varphi \leftrightarrow \psi)$.*

So, for every $\mathcal{L}_{T,s}$-claim there must be a $\mathcal{L}_T$-claim that is T'-equivalent in order to satisfy eliminability of s.

The constraint of non-creativity/conservativity can be characterized as follows:

Criterion 2 (Non-creativity) *T' is a non-creative extension of T iff for all $\varphi \in \mathcal{L}$ it holds: $\vdash^{T'} \varphi$ iff $\vdash^{T} \varphi$.*

So, according to this constraint, usages of the old vocabulary remain unchanged.

Now it can be shown that explicit definitions and only explicit definitions satisfy both constraints. Since reductions and characterizations of symbols sometimes only succeed partially, one often allows for language conventions also partial definitions (as, e.g., is the case for arithmetics, where the operator for division is defined only partially for denominators $\neq 0$). Such partial definitions still satisfy the constraint for non-creativity, but not for full (only partial) eliminability. We will allow also for partial characterizations here with the demand that the introduction of a symbol by multiple characterization first has to be non-creative and second, every further characterization has to increase eliminiability in such a way that the set of eliminable statements is a proper superset of the foregoing characterization. Note that this allows also for circular characterizations since such characterizations also can increase the eliminability of a symbol.

Having indicated the conventional framework, we can go on with a conventional reconstruction of Belnap's basic system.

3 Logic by Conventions

In this section we just provide the conventional reconstruction of Belnap's basic system. It is as follows—since the aforementioned conventional rules are mainly studied with respect to relational symbols, we also reconstruct the inference relation by a set of n-ary relation symbols ($\mathcal{R}^n$ where one may have classically in mind: $X \vdash \varphi$ iff there is a n such that $n = \#X$ and $X_1, \ldots, X_n \mathcal{R}^n \varphi$):

- Conventional characterization of $\mathcal{R}^1$:

  Ax1 If $x = y$, then $x\mathcal{R}^1 y$ (Reflexivity)

  Ax2 $x\mathcal{R}^1 z$ iff there is a y such that $x\mathcal{R}^1 y$ and $y\mathcal{R}^1 z$ (Transitivity)

- Conventional characterization of $\mathcal{R}^2$:

  Ax3 If $x = y$, then $x, y\mathcal{R}^2 z$ iff $x\mathcal{R}^1 z$ (Contraction)

  Ax4 If $x\mathcal{R}^1 z$, then $x, y\mathcal{R}^2 z$ (Weakening)

  Ax5 If $w\mathcal{R}^1 z$, then: If $x, y\mathcal{R}^2 w$, then $x, y\mathcal{R}^2 z$ (Transitivity$_{Right}$)

  Ax6 $x, y\mathcal{R}^2 z$ iff there is a w sucht that $x\mathcal{R}^1 w$ and $y, w\mathcal{R}^2 z$ (Permutability and Transitivity$_{Left}$)

- Explicit definition of &:

  Ax7 $x\&y =_{\mathcal{R}} z$ iff

  - If $x = y$, then $z = x$

- If $x \neq y$, then:
  - $z\mathcal{R}^1 x$ and $z\mathcal{R}^1 y$ and $x, y\mathcal{R}^2 z$ and
  - for all w it holds that: If $w\mathcal{R}^1 x$ and $w\mathcal{R}^1 y$, then $w\mathcal{R}^1 z$

- Explicit definition of $\mathcal{R}^n$ ($n \geq 3$):

  Ax9 $x_1, \ldots, x_n\mathcal{R}^n z$ iff $x_1\& \ldots \& x_n\mathcal{R}^1 z$

- [20] Explicit definition of $\perp$:

  Ax8 $\perp =_{\mathcal{R}} z$ iff for all x it holds that $z\mathcal{R}^1 x$

⋮

Some remarks: All characterizations are non-creative w.r.t. the foregoing characterizations. I.e. $\{$Ax1–Ax2$\}$ w.r.t. $\emptyset$; $\{$Ax3–Ax6$\}$ w.r.t. $\{$Ax1–Ax2$\}$ and so forth. Furthermore, all multiple characterizations increase eliminability: Ax1,Ax3,Ax4 are multiple partial definitions; Ax2,Ax5,Ax6 allow for the following theorems:

Th1 For all x, y, z it holds that: $x, y\mathcal{R}^2 z$ iff $y, x\mathcal{R}^2 z$

Th2 For all x, y, z_1, z_2 it holds that: If $z_1\mathcal{R}^1 z_2$ and $z_2\mathcal{R}^1 z_1$, then:

- $z_1\mathcal{R}^1 x$ iff $z_2\mathcal{R}^1 x$
- $x\mathcal{R}^1 z_1$ iff $x\mathcal{R}^1 z_2$
- $z_1, y\mathcal{R}^2 x$ iff $z_2, y\mathcal{R}^2 x$
- $y, z_1\mathcal{R}^2 x$ iff $y, z_2\mathcal{R}^2 x$
- $x, y\mathcal{R}^2 z_1$ iff $x, y\mathcal{R}^2 z_2$

Th3 For all x, y it holds that $x\&y =_{\mathcal{R}} y\&x$

Th4 For all x_1, x_2, y it holds that: If $x_1\mathcal{R}^1 x_2$ and $x_2\mathcal{R}^1 x_1$, then $x_1\&y =_{\mathcal{R}} x_2\&y$

Th5 For all x, y, z_1, z_2 it holds that: If $x\&y =_{\mathcal{R}} z_1$ then: $x\&y =_{\mathcal{R}} z_2$ iff $z_1\mathcal{R}^1 z_2$ and $z_2\mathcal{R}^1 z_1$

Th6 For all x, y, z it holds that: $x\&(y\&z) =_{\mathcal{R}} (x\&y)\&z$

Furthermore it holds: The definitional characterisation is correct (replacing the $\mathcal{R}$s by $\vdash$ results in correct rules). It is also complete (w.r.t. propositional logic *pl*; due to compactness of *pl* we can re-write every *pl*-proof in our $\mathcal{R}$-notation; for all $\mathcal{R}$s there hold the respective structural rules).

4 Conclusion

We have argued here for a conventional foundation of logic by joining Belnap's approach of justifying a distinction of the vocabulary of an artificial language into a logical and a non-logical one based on the "explainabilty" of such symbol's usage by help of basic rules alone. As a conventional framework we have choosen a strongly weakened form of explicit definability that allows for partial characterization (also with circularity). We have presented a conventional reconstruction of a basic logical system and by this indicated a conventional foundation of logic.

References

Belnap, Nuel (1962). "Tonk, Plonk and Plink". In: *Analysis* 22.6, pp. 130–134.

Frege, Gottlob (1884). *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*. Breslau: Verlag von Wilhelm Koebner.

Quine, Willard van Orman (1986). *Philosophy of Logic*. Second Edition. Harvard University Press.

Tarski, Alfred (1986). "What are Logical Notions?" In: *History and Philosophy of Logic* 7.2. Ed. by Corcoran, John, pp. 143–154.